Stagnation and Innovation Before Agriculture

November 2008

Appendix: Proofs of Propositions

Proof of Proposition 1.

Choose any resource r. Let $Z \equiv X(n_r^t/N^t)N^{t+1} \equiv Xn_r^t\rho^t$ be the number of copies of string k_r^t, where X is the number of observations per child, n_r^t/N^t is the fraction of these observations that pertain to resource r, and N^{t+1} is the number of children who survive to become adults in period t+1. Let z = 1 . . Z index the individual copies k_{rz} of k_r^t.

The number of digits in k_r^t that match k_r^* is $q_r^t \in \{0, 1 .. Q\}$ and the number of digits in k_{rz} that match k_r^* is $q_{rz} \in \{0, 1 .. Q\}$. The random variables q_{rz} are iid conditional on k_r^t. Fix the string k_r^t and define the probabilities

$$\pi_q(p) \equiv \Pr(q_{rz} = q \mid k_r^t; p) \qquad \text{for all } q = 0, 1 .. Q \text{ and } z = 1 .. Z.$$

Then define

$$\theta_q(p) \equiv \Pr(q_r^{t+1} = q \mid k_r^t; p) = \Pr(\max\{q_{r1} .. q_{rZ}\} = q \mid k_r^t; p)$$

$$= \Pr(q_{rz} \leq q \text{ for } z = 1 .. Z \mid k_r^t; p) - \Pr(q_{rz} \leq q\text{-}1 \text{ for } z = 1 .. Z \mid k_r^t; p)$$

$$= [\pi_0(p) + .. + \pi_q(p)]^Z - [\pi_0(p) + .. + \pi_{q-1}(p)]^Z$$

where $\pi_0(p) + .. + \pi_{q-1}(p) \equiv 0$ for q = 0.

We want to compute each $\theta_q(p)$ when $X \to \infty$ and $p \to 0$ such that $Xp \equiv \lambda > 0$ is constant. From the definition of Z we have $Z(p) = \lambda\rho^t n_r^t/p$. In what follows we drop the r subscript and the t superscripts in this expression. Next define

$$\theta_q^* \equiv \lim_{p\to 0} \theta_q(p)$$

$$= \lim_{p\to 0} [\pi_0(p) + .. + \pi_q(p)]^{Z(p)} - \lim_{p\to 0} [\pi_0(p) + .. + \pi_{q-1}(p)]^{Z(p)}$$

$$= \{\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + .. + \pi_q(p))]\}^{\lambda n\rho}$$

$$- \{\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + .. + \pi_{q-1}(p))]\}^{\lambda n\rho}$$

We have $\lim_{p\to 0} \pi_q(p) = 1$ for $q = q_r^t$ and $\lim_{p\to 0} \pi_q(p) = 0$ for $q \neq q_r^t$ because at least one mutation must occur whenever the number of correct digits differs from q_r^t. This implies that for each expression of the form $\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + . . + \pi_q(p))]$, there are two possible cases:

(a) If $q < q_r^t$ then $\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + . . + \pi_q(p))] = e^{-\infty} = 0$ and hence $\theta_q^* = 0$.

(b) If $q \geq q_r^t$ then $\lim_{p\to 0} (\pi_0(p) + . . + \pi_q(p)) = 1$ and hence $\lim_{p\to 0} \exp[(1/p)\ln(\pi_0(p) + . . \pi_q(p))] = \lim_{p\to 0} \exp[\pi_0'(p) + . . + \pi_q'(p)]$.

Now consider the derivatives $\pi_q'(p)$ in case (b) above. The probabilities $\pi_q(p)$ are polynomials in p, and all outcomes involving two or more mutations correspond to terms that are quadratic or higher. After taking derivatives, all such terms vanish in the limit. Thus we can confine attention to outcomes that involve either no mutations or just one mutation. This implies that only $q = q_r^t - 1$, $q = q_r^t$, and $q = q_r^t + 1$ are relevant.

(i) $q = q_r^t - 1$. This outcome can be obtained in q_r^t ways by having one mutation at a correct locus and no mutations elsewhere, which has the probability $q_r^t p(1-p)^{Q-1}$. All other ways to obtain the same result involve three or more mutations. This gives $\lim_{p\to 0} \pi_q'(p) = q_r^t$ so $\lim_{p\to 0} \exp[\pi_0'(p) + . . + \pi_q'(p)] = \exp(q_r^t)$.

(ii) $q = q_r^t$. This outcome can be obtained in one way with no mutations, which has probability $(1-p)^Q$. All other ways to obtain the same result involve two or more mutations. This gives $\lim_{p\to 0} \pi_q'(p) = -Q$ so $\lim_{p\to 0} \exp[\pi_0'(p) + . . + \pi_q'(p)] = \exp[-(Q-q_r^t)]$.

(iii) $q = q_r^t + 1$. This outcome can be obtained in $Q-q_r^t$ ways by having one mutation at an incorrect locus and no mutations elsewhere, with probability $(Q-q^t)p(1-p)^{Q-1}$. All other ways to obtain the same result involve three or more mutations. This gives $\lim_{p\to 0} \pi_q'(p) = Q-q_r^t$ and thus $\lim_{p\to 0} \exp[\pi_0'(p) + . . + \pi_q'(p)] = e^0 = 1$.

Recall from (a) above that $\theta_q^* = 0$ when $q < q_r^t$. To solve for θ_q^* when $q = q_r^t$ we first observe that the second limit in the last line for θ_q^* is zero due to (a). Substituting from (b) and (ii) in the first limit in the last line for θ_q^* gives $\theta_q^* = \exp[-\lambda n_r^t \rho^t (Q-q^t)]$.

To solve for θ_q* when $q = q_r^t + 1$ we observe that (b) applies to both limits in the last line for θ_q*. Substituting from (b) gives $\theta_q^* = 1- \exp[-\lambda n_r^t \rho^t (Q-q^t)]$.

Finally, we have $\theta_q^* = 0$ when $q \geq q_r^t + 2$ because (b) applies to both limits in the last line for θ_q* and both of these limits equal unity. The latter result follows from (iii) and the fact that any outcome with $q \geq q_r^t + 2$ requires two or more mutations.

By construction all of the θ_q* are conditional on k_r^t. However, the structure of the proof shows that the only relevant property of k_r^t is the number of correct digits q_r^t. Thus we can write the transition probabilities for q_r^{t+1} and q_r^r as in Proposition 1.

The limiting transition probabilities for strings, $\lim_{p\to 0} \Pr(k_r^{t+1} = k \mid k_r^t; p)$, follow from the preceding results. If k has fewer correct digits than q_r^t or more than $q_r^t + 1$, it has probability zero in the limit. The only way to have $q_r^{t+1} = q_r^t$ in the limit is by having $k_r^{t+1} = k_r^t$ so this has probability $\exp[-\lambda n_r^t \rho^t (Q-q^t)]$. There are $Q-q_r^t$ ways to obtain $q_r^{t+1} = q_r^t + 1$ by a single mutation that changes one incorrect digit to a correct one. Each of these strings k_r^{t+1} has probability $\{1- \exp[-\lambda n_r^t \rho^t (Q-q^t)]\}/(Q-q_r^t)$. This completes the proof.

Proof of Proposition 2.

Uniqueness follows from the strict concavity of the objective function and continuity follows from the theorem of the maximum.

(a) Let n' be optimal for (A, N') and let n'' be optimal for (A, N''). Suppose that $n_r'' \le n_r'$. The first order conditions for (2) with parameters (A, N'') imply that all s with $n_s'' > 0$ have $A_s f_s'(n_s'') \ge A_r f_r'(n_r'')$. The fact that $n_r'' \le n_r'$ gives $A_r f_r'(n_r'') \ge A_r f_r'(n_r')$. Finally, the first order conditions for (2) with parameters (A, N') and $n_r' > 0$ give $A_r f_r'(n_r') \ge A_s f_s'(n_s')$ for all $s = 1 \,.\,.\, R$. This series of inequalities gives $A_s f_s'(n_s'') \ge A_s f_s'(n_s')$ for all s with $n_s'' > 0$ and thus implies $n_s' \ge n_s''$ for all s such that $n_s'' > 0$. Clearly $n_s' \ge n_s''$ also holds for all s such that $n_s'' = 0$. Summing over resources gives $N' \ge N''$, contradicting the assumption $N' < N''$. This shows that $n_r'' > n_r'$.

(b) Let n' be optimal for (A', N) and let n'' be optimal for (A'', N). Suppose $n_r'' \le n_r'$. For all $v \ne r$ such that $n_v'' > 0$ we have $A_v'' f_v'(n_v'') \ge A_r'' f_r'(n_r'')$. Furthermore, $A_r'' f_r'(n_r'') > A_r' f_r'(n_r'') \ge A_r' f_r'(n_r') \ge A_v' f_v'(n_v')$ for all $v = 1 \,.\,.\, R$. The first inequality follows from $A_r'' > A_r'$, the second from $n_r'' \le n_r'$, and the last from $n_r' > 0$. The preceding series of inequalities and $A_v'' = A_v'$ for $v \ne r$ shows that $A_v' f_v'(n_v'') > A_v' f_v'(n_v')$ for all $v \ne r$ such that $n_v'' > 0$ and hence $n_v'' < n_v'$ for all $v \ne r$ such that $n_v'' > 0$. There must be at least one such $v \ne r$ since otherwise $n_r'' = N > n_r'$ due to $n_s' > 0$, but we have supposed $n_r'' \le n_r'$. Clearly all $v \ne r$ with $n_v'' = 0$ have $n_v'' \le n_v'$. Summing over all resources gives $N'' < N'$ because there is at least one $v \ne r$ with $n_v'' < n_v'$. This contradicts the fact that N is constant. Therefore $n_r'' > n_r'$. Next suppose $n_s'' \ge n_s'$. Consider $v \ne s$ and $v \ne r$. For all such v, $A_v'' f_v'(n_v'') \le A_s'' f_r'(n_s'') \le A_s'' f_s'(n_s') = A_s' f_s'(n_s')$. Moreover, if $n_v' > 0$ we have $A_s' f_s'(n_s') = A_v' f_v'(n_v')$. This and $A_v'' = A_v'$ implies that for any $v \ne s$ and $v \ne r$ with $n_v' > 0$, it must be true that $n_v'' \ge n_v'$. Clearly the same inequality holds when $n_v' = 0$. Since

$n_r'' > n_r'$ and we have supposed $n_s'' \geq n_s'$, summing over resources gives $N'' > N'$. This contradicts the fact that N is constant. Therefore $n_s'' < n_s'$.

(c) Fix $A > 0$. Choose any $N' \neq N''$ and $\mu \in (0, 1)$. Let n′ be optimal for (A, N′) and let n″ be optimal for (A, N″). Define $n^\mu = \mu n' + (1-\mu)n'' \geq 0$. This is a feasible allocation for the total population $N^\mu = \mu N' + (1-\mu)N''$. It follows that $H(A, N^\mu) \geq \sum A_r f_r(n_r^\mu) = \sum A_r f_r[\mu n_r' + (1-\mu)n_r''] > \sum A_r[\mu f_r(n_r') + (1-\mu)f_r(n_r'')] = \mu H(A, N') + (1-\mu)H(A, N'')$. The strict inequality occurs because due to the strict concavity of f_r we have $f_r[\mu n_r' + (1-\mu)n_r''] > \mu f_r(n_r') + (1-\mu)f_r(n_r'')$ whenever $n_r' \neq n_r''$, and the latter inequality must hold for at least one r because $N' \neq N''$. Due to $H(A, 0) = 0$, the strict concavity of H implies $H(A, \mu N) > \mu H(A, N)$ for all $N > 0$ and $\mu \in (0, 1)$. This yields $H(A, \mu N)/\mu N > H(A, N)/N$ for all $N > 0$ and $\mu \in (0, 1)$. Thus $y(A, N) \equiv H(A, N)/N$ is decreasing in N.

(d) Fix $A > 0$ and consider the (unique) optimal allocation n(A, N). We first show that $\lim_{N\to\infty} n_s(A, N) = \infty$ must hold for some s. Suppose instead that for every r there is a finite upper bound $\underline{n}_r$ such that $n_r(A, N) \leq \underline{n}_r$ for all N. Then for any $N > \sum \underline{n}_r$ we have $\sum n_r(A, N) < N$, which contradicts optimality. Thus there is some s such that $\lim_{N\to\infty} n_s(A, N) = \infty$. From the assumption that $f_r'(n_r) \to 0$ as $n_r \to \infty$ for r = 1 . . R we obtain $\lim_{N\to\infty} A_s f_s'[n_s(A, N)] = 0$. Next, define $m(A, N) = \max \{A_r f_r'[n_r(A, N)]\}$. There is some $\underline{N}$ such that $N > \underline{N}$ implies $n_s(A, N) > 0$. From the first order conditions for (2) this implies $m(A, N) = A_s f_s'[n_s(A, N)]$ for all $N > \underline{N}$. Hence $\lim_{N\to\infty} m(A, N) = 0$, which implies $\lim_{N\to\infty} A_r f_r'[n_r(A, N)] = 0$ for all r = 1 . . R. Thus $\lim_{N\to\infty} n_r(A, N) = \infty$ for all r = 1 . . R. From part (c), H(A, N)/N is decreasing in N. Suppose that this ratio has a lower bound $\delta > 0$. This implies $\sum \{A_r f_r[n_r(A, N)] - \delta n_r(A, N)\} \geq 0$. However, we have $f_r(n_r)/n_r \to 0$ as $n_r \to \infty$ for r = 1 . . R. This is obvious if there is a finite upper bound on $f_r(n_r)$. If $f_r(n_r)$ is unbounded then using $f_r'(n_r) \to 0$ as $n_r \to \infty$ gives the same result. The facts that $\lim_{N\to\infty} n_r(A, N) = \infty$ and

$f_r(n_r)/n_r \to 0$ as $n_r \to \infty$ for all $r = 1 \,.\,. R$ together imply that there is some sufficiently large N such that $A_r f_r[n_r(A, N)] - \delta n_r(A, N) < 0$ for all $r = 1 \,.\,. R$. This contradicts the earlier inequality and gives the desired result $\lim_{N\to\infty} H(A, N)/N = 0$.

(e) Fix $A > 0$. By the envelope theorem $H(A, N)$ is differentiable in N and $H_N(A, N)$ is the Lagrange multiplier for (2). Since $H(A, 0) = 0$, $\lim_{N\to 0} H(A, N)/N = \lim_{N\to 0} H_N(A, N)$. When $N > 0$, the first order conditions for (2) give $H_N(A, N) = \max \{A_r f_r'[n_r(A, N)]\}$. Because $n_r(A, N)$ is continuous in N with $n_r(A, 0) = 0$ for all r, we have $\lim_{N\to 0} H_N(A, N) = \max \{A_r f_r'(0)\}$.

Proof of Proposition 3.

Sufficiency. (K^S, N^S, n^S) is a VLRE if (i) $k_r^S = k_r^*$ for all r such that $n_r^S > 0$; (ii) $N^S = N[A(a, K^S)]$ is derived from (4); and (iii) $n^S = n[A(a, K^S), N^S]$ is derived from (2). The second and third conditions hold by construction. Condition (i) holds if $n_r^S = 0$ for all $r \notin S$. From the first order conditions for (2), this is true if $H_N(A^S, N^S) \geq A_r(k_r^{min})f_r'(0)$ for all $r \notin S$, which is true from (b) in Proposition 3. $N^S > 0$ holds if $A_r^S f_r'(0) > y^*$ for some $r \in \{1 .. R\}$, which is true from (a) in Proposition 3. This shows that (K^S, N^S, n^S) is a non-null VLRE of type S. Now suppose (K', N', n') is some other non-null VLRE of type S. This implies $H(A^S, N^S)/N^S = H(A', N')/N' = y^*$. By the construction of A^S, we have $A' \geq A^S$ where these vectors differ at most for $r \notin S$. By the definition of VLRE, $n_r' = 0$ for all $r \notin S$. Reducing the productivities of one or more resources that are not in use has no effect on $H(A, N)$ and therefore $H(A', N') = H(A^S, N')$. This implies $H(A', N')/N' = H(A^S, N')/N' = y^*$, which in turn gives $N' = N^S$. The uniqueness of the solution in (2) then gives $n' = n^S$.

Necessity. Suppose (K', N', n') is a non-null VLRE of type S but (a) in Proposition 3 does not hold. The definition of VLRE implies $n_r' = 0$ for all $r \notin S$. Setting $A' = A(a, K')$, we have $\max \{A_r' f_r'(0)\} = H_N(A', 0) > H(A', N')/N' > H_N(A', N') \geq A_r' f_r'(0)$ for all $r \notin S$. The equality follows from Proposition 2(e), the two strict inequalities follow from $N' > 0$ and Proposition 2(c), and the weak inequality follows from $n_r' = 0$ for $r \notin S$ and the first order conditions for (2). This series of results implies that $A_r' f_r'(0) = H_N(A', 0)$ for some $r \in S$ because otherwise there is a contradiction. But $H(A', N')/N' = y^*$ from (4) because (K', N', n') is a non-null VLRE. Moreover, $y^* \geq A_r' f_r'(0)$ because (a) in Proposition 3 does not hold. Again this leads to a contradiction. Therefore if a non-null VLRE of type S exists, condition (a) in Proposition 3 must hold.

Now suppose (K', N', n') is a non-null VLRE of type S and (a) in Proposition 3 holds, but (b) in Proposition 3 does not. The definition of VLRE implies $n_r' = 0$ for all $r \notin$

S. From the first order conditions for (2), this implies $H_N(A', N') \geq A_r(k_r')f_r'(0)$ for all $r \notin$ S. Moreover, $A_r(k_r')f_r'(0) \geq A_r(k_r^{min})f_r'(0)$ for all $r \notin S$ by the definition of k_r^{min}, and $A_r(k_r^{min})f_r'(0) > H_N(A^S, N^S)$ holds for some $r \notin S$ because (b) in Proposition 3 is violated. This implies $H_N(A', N') > H_N(A^S, N^S)$. But this cannot be true because the only possible difference between A' and A^S is $A_r' > A_r^S$ for one or more $r \notin S$. Reducing productivity for a resource with $n_r(A', N') = 0$ has no effect on the optimal labor allocation or on the maximum value in (2). This implies $H(A', N')/N' = H(A^S, N')/N' = y^*$ and therefore $N' = N^S$. It follows that $H_N(A', N') = H_N(A^S, N^S)$. This contradiction shows that if a non-null VLRE of type S exists, condition (b) in Proposition 3 must hold.

Proof of Proposition 4.

(a) Consider any sample path $\{K^t, N^t\}$ for $t \geq 0$. There are finitely many possible repertoires, so at least one repertoire K' must be repeated infinitely many times. Because the sequence of productivity vectors $\{A^t\} = \{A(K^t)\}$ is non-decreasing by Proposition 1 and conservation of latent strings, it is impossible to return to an earlier repertoire after departing from it. Therefore only one repertoire can occur infinitely many times, and the occurrences of this K' must be consecutive. Let T be the first period in which K' occurs. Using $N^0 > 0$ and $\rho(y) > 0$ for $y > 0$, (3) implies $N^T > 0$. Due to $K^t = K'$ for all $t \geq T$, the condition MPA in section 4 gives $\{N^t\} \rightarrow N'$. The result $\{n^t\} \rightarrow n'$ follows from the continuity of solutions in (2).

(b) Suppose some terminal array (K', N', n') is not a VLRE. This implies that $n' = n[A(K'), N']$ has $n_r' > 0$ for some r with $k_r' \neq k_r^*$. Define M by $H_N(A', M) \equiv \max \{a_r g_r(k_r')f_r'(0)$ for r such that $k_r' \neq k_r^*\}$. Given the terminal productivities A', M is the largest population such that all improvable techniques are latent. A unique $M \in [0, \infty)$ exists because Proposition 2(e) gives $H_N(A', 0) = \max \{a_r g_r(k_r')f_r'(0)$ for $r = 1 .. R\} \geq \max \{a_r g_r(k_r')f_r'(0)$ for r such that $k_r' \neq k_r^*\}$; Proposition 2(d) gives $H_N(A', \infty) = 0 < \min \{a_r g_r(k_r')f_r'(0)$ for r such that $k_r' \neq k_r^*\}$; and $H_N(A', N)$ is continuous and decreasing in N. Since (K', N', n') is not a VLRE, an improvable technique must be active in n' and therefore $M < N'$.

From (a), for each sample path having K' as the terminal repertoire there is some $T \geq 0$ such that $K^t = K'$ for all $t \geq T$. Consider any such sample path. There are two possibilities: (i) $N^T \in (0, M]$ or (ii) $N^T \in (M, \infty)$. In case (i), $K^t = K'$ for all $t \geq T$, $N^T \leq M < N'$, and MPA imply that after finitely many periods we must have $N^\tau \in (M, \infty)$. Setting $T = \tau$ if necessary, it therefore suffices to consider (ii). In this case, MPA ensures that if $N^T < N'$ then $N^t \geq N^T$ for all $t \geq T$, and if $N' \leq N^T$ then $N^t \geq N'$ for all $t \geq T$. Thus $M < \min \{N', N^T\} \leq N^t$ for all $t \geq T$.

The constancy of A' for $t \geq T$, the scale effect in Proposition 2(a), and the construction of M ensure that for some r with $k_r' \neq k_r^*$ there is a lower bound $\underline{n}_r$ such that $0 < \underline{n}_r \leq n_r^t$ for all $t \geq T$. Next, define $\rho^t \equiv N^{t+1}/N^t$ as in Proposition 1. MPA ensures that if $N^T \leq N'$ then $\rho^t \geq \underline{\rho} \equiv 1$ for all $t \geq T'$ and if $N^T > N'$ then $\rho^t \geq \underline{\rho} \equiv N'/N^T > 0$. Using the lower bounds $\underline{n}_r$ and $\underline{\rho}$ along with $k_r' \neq k_r^*$, Proposition 1 implies that for each $t \geq T$ the probability that k_r stays unchanged cannot exceed $\exp(-\lambda\underline{\rho}\underline{n}_r) < 1$. Over the unbounded interval $t \geq T$, the probability that k_r remains unchanged vanishes. Therefore $\Pr(K^t = K' \text{ for } t \geq T \mid K^T = K', N^T) = 0$.

Every sample path has a terminal repertoire K' (which may or may not generate a VLRE) and the number of repertoires is finite, so we can partition the set of sample paths starting from (K^0, N^0) into finitely many sets indexed by K'. The probability of a particular terminal repertoire K' is Prob(K' is terminal) =

$$\sum_{T \in \{0, 1 \ldots\}} \sum_{\underline{N} \in N(K',T)} \Pr(K^T = K', N^T = \underline{N} | K^0, N^0) \Pr(K^t = K' \text{ for } t \geq T | K^T = K', N^T = \underline{N})$$

where T is the first date on which K' occurs and $N(K', T)$ is the set of population levels that are consistent with a first occurrence of K' in period T. For any finite T there are finitely many possible mutation histories, and each of these histories determines a unique N^T, so $N(K', T)$ is a finite set. Moreover, $N^0 > 0$ implies $N^T > 0$ for any finite T from (3). We have shown that if the terminal array (K', N', n') is not a VLRE then $\Pr(K^t = K' \text{ for } t \geq T | K^T = K', N^T = \underline{N}) = 0$ for all $\underline{N} > 0$ and all $T \geq 0$. Thus the probability that K' is the terminal repertoire is zero if K' does not generate a VLRE. Since there are finitely many terminal repertoires, the terminal array (K', N', n') is a VLRE with probability one.

(c) Suppose $0 < N^t < N[A(K^t)]$. From MPA, Proposition 1, the conservation of latent strings, and the fact that N(A) is non-decreasing we have $N^t < N^{t+1} < N[A(K^t)] \leq N[A(K^{t+1})]$. When $0 < N^0 < N[A(K^0)]$, we can repeat the argument to obtain $0 < N^t < N^{t+1}$ for all $t \geq 0$. When $0 < N^0 = N[A(K^0)]$, we have $N^t = N[A(K^0)]$ for $0 \leq t \leq T$

where T is the first period (if any) at which a mutation occurs for an active resource. This yields $N^0 = N^T < N[A(K^T)]$. For $t \geq T$, $\{N^t\}$ is increasing as before. When $N^0 > N[A(K^0)] \geq 0$, MPA ensures $N^0 > N^1 > N[A(K^0)]$. If there is a period $T \geq 1$ in which a mutation to an active resource yields $N^T \leq N[A(K^T)]$ then $\{N^t\}$ is non-decreasing for $t \geq T$ by the reasoning used above (and increasing if $N^T < N[A(K^T)]$ holds). Otherwise, we have $N^t > N[A(K^t)]$ for all $t \geq 0$. From MPA this implies $N^t > N^{t+1} > N[A(K^t)]$ for all $t \geq 0$ and $\{N^t\}$ is decreasing.

Proof of Proposition 5.

(a) Let the climate change permanently to a' at the start of period $t = 0$ before labor is allocated. The repertoire and population (K^0, N^0) at $t = 0$ are inherited from the previous VLRE. Due to the neutrality of the shock, the optimal labor allocation in (2) for $t = 0$ is unaffected (the solution $n(A, N)$ is homogeneous of degree zero in A). We use n^0 interchangeably for the labor allocation in the original VLRE and the allocation in period $t = 0$ after the climate change occurs. The productivity vector in period $t = 0$ is $\theta A^0 = \theta A(a^0, K^0) = A(\theta a^0, K^0)$ and the corresponding population in LRE is denoted by $N' = N(\theta A^0)$.

We will show that for any $T \geq 0$, $K^T = K^0$ and $N^T \in (N', N^0]$ implies $K^{T+1} = K^0$ and $N^{T+1} \in (N', N^0]$. This implies that the system converges to a VLRE with $K' = K^0$ and $N' = N(\theta A^0)$.

The labor allocation associated with (K^T, N^T) is $n^T = n(\theta A^0, N^T)$. Two necessary conditions for $k_r^{T+1} \neq k_r^T$ are (i) $k_r^T \neq k_r*$ and (ii) $n_r^T > 0$. Any r satisfying (ii) has $n_r(\theta A^0, N^T) > 0$ and hence $n_r(\theta A^0, N^0) > 0$ from Proposition 2(a) and $N^0 \geq N^T$. But then $n_r(A^0, N^0) > 0$ by the homogeneity of $n(A, N)$ in A. This implies $k_r^0 = k_r^T = k_r*$ because (K^0, N^0, n^0) is a VLRE for climate a^0. Thus (i) cannot hold. This shows that $k_r^{T+1} = k_r^T$ for all r and hence $K^{T+1} = K^0$. We also have $H(\theta A^0, N')/N' = y* > H(\theta A^0, N^T)/N^T$ because $N^T > N'$. Therefore by MPA, $N^{T+1} \in (N', N^T) \subseteq (N', N^0]$. This establishes deterministic convergence to a unique VLRE (K', N', n') such that (i) $K' = K^0$ and (ii) $N' = N(\theta A^0) < N^0$.

Suppose $n_r^0 = n_r(A^0, N^0) = 0$. If $n_r' = n_r(\theta A^0, N') > 0$ then $n_r(A^0, N') > 0$ by homogeneity. Furthermore, $n_r(A^0, N^0) > 0$ from $N' < N^0$ and Proposition 2(a), which is false. Therefore $n_r' = 0$. This shows that the active resources in n' are a subset of the active resources in n^0.

Starting from the (non-null) VLRE (K', N', n') associated with climate a', suppose in period $t = 0$ the climate returns permanently to $a^0 = a'/\theta$. We will show that for any $T \geq 0$, $K^T = K'$ and $N^T \in [N', N^0)$ implies $K^{T+1} = K'$ and $N^{T+1} \in [N', N^0)$. It follows that the system converges to (K^0, N^0).

Using $K^T = K' = K^0$, the labor allocation associated with (K^T, N^T) is $n^T = n(A^0, N^T)$. As before, two necessary conditions for $k_r^{T+1} \neq k_r^T$ are (i) $k_r^T \neq k_r^*$ and (ii) $n_r^T > 0$. Any r satisfying (ii) has $n_r(A^0, N^T) > 0$ and hence $n_r(A^0, N^0) > 0$ from $N^0 > N^T$ and Proposition 2(a). But every r with $n_r^0 = n_r(A^0, N^0) > 0$ has $k_r^0 = k_r^T = k_r^*$ because (K^0, N^0, n^0) is a VLRE for a^0 and $K^T = K' = K^0$. Thus (i) cannot hold. This shows that $k_r^{T+1} = k_r^T$ for all r and therefore $K^{T+1} = K'$. We also have $H(A^0, N^0)/N^0 = y^* < H(A^0, N^T)/N^T$ because (K^0, N^0, n^0) is a VLRE for a^0 and $0 < N^T < N^0$. Therefore by MPA, $N^{T+1} \in [N^T, N^0) \subseteq [N', N^0)$. This shows deterministic convergence to the original VLRE (K^0, N^0, n^0).

(b) Suppose the terminal array (K', N', n') is a VLRE as in Proposition 4(b). From the fact that (K^0, N^0, n^0) is a VLRE for the climate a^0 and $H(A, N)$ is linearly homogeneous in A, $y^* = H(A^0, N^0)/N^0 < H(\theta A^0, N^0)/N^0$. This shows that $N^0 < N(\theta A^0)$. Using Proposition 4(c), the latter inequality implies $\{N^t\}$ is increasing and therefore $N^0 \leq N^t < N'$ for all $t \geq 0$.

<u>Necessity</u>. Suppose $n_r[\theta A^0, N(\theta A^0)] = 0$ for all r such that $k_r^0 \neq k_r^*$. We will show that $K^T = K^0$ and $N^T \in [N^0, N(\theta A^0))$ implies $K^{T+1} = K^0$ and $N^{T+1} \in [N^0, N(\theta A^0))$. Repeating the argument then yields $K' = K^0$ and $N' = N(\theta A^0)$. Two necessary conditions for $k_r^{T+1} \neq k_r^T$ are (i) $k_r^T \neq k_r^*$ and (ii) $n_r^T > 0$. From $N^T < N(\theta A^0)$ and Proposition 2(a), (ii) implies $n_r[\theta A^0, N(\theta A^0)] > 0$. Our initial supposition and $K^T = K^0$ imply that (i) is false. Thus $k_r^{T+1} = k_r^T$ for all r and $K^{T+1} = K^T$. We have $H[\theta A^0, N(\theta A^0)]/N(\theta A^0) = y^* < H(\theta A^0, N^T)/N^T$ from the definition of $N(\theta A^0)$ and the fact

that $N^T < N(\theta A^0)$. Therefore by MPA, $N^{T+1} \in (N^T, N(\theta A^0)) \subseteq [N^0, N(\theta A^0))$ as claimed.

Sufficiency. Suppose $n_r[\theta A^0, N(\theta A^0)] > 0$ for some r such that $k_r^0 \neq k_r^*$, but $K' = K^0$. The conditions for VLRE require $H(\theta A^0, N')/N' = y^*$ so $N' = N(\theta A^0)$. The conditions for VLRE also require $n_r' = n_r[\theta A^0, N(\theta A^0)] = 0$ for all r such that $k_r' \neq k_r^*$. According to our supposition this is false. Therefore $K' \neq K^0$.

We have already shown that if (*) does not hold then $N' = N(\theta A^0)$. Suppose that (*) does hold. We need to show that $N' > N(\theta A^0)$. Define $A' \equiv A(a', K')$. Because (K', N', n') is a VLRE for the climate a′ we have $H(A', N')/N' = y^* = H[\theta A^0, N(\theta A^0)]/N(\theta A^0)$. We cannot have $N' < N(\theta A^0)$ since then $A' \geq \theta A^0$ gives $y^* = H[\theta A^0, N(\theta A^0)]/N(\theta A^0) < H(\theta A^0, N')/N' \leq H(A', N')/N' = y^*$, which is a contradiction. Therefore to show $N' > N(\theta A^0)$ it suffices to rule out $N' = N(\theta A^0)$. Assume $N' = N(\theta A^0)$ holds. It follows from $H(A', N')/N' = y^* = H[\theta A^0, N(\theta A^0)]/N(\theta A^0)$ that $H(A', N') = H[\theta A^0, N(\theta A^0)]$. Consider the case $n' \neq n[\theta A^0, N(\theta A^0)]$. From the uniqueness of solutions in (2) and the fact that both of the labor allocations involved are feasible, we have $H(A', N') = \sum \theta a_r^0 g_r(k_r') f_r(n_r') > \sum \theta a_r^0 g_r(k_r') f_r[n_r(\theta A^0, N')] \geq \sum \theta a_r^0 g_r(k_r^0) f_r[n_r(\theta A^0, N')] = H[\theta A^0, N(\theta A^0)]$. This contradicts $H(A', N') = H[\theta A^0, N(\theta A^0)]$. Next consider the case $N' = N(\theta A^0)$ and $n' = n[\theta A^0, N(\theta A^0)]$. Now $H(A', N') = H[\theta A^0, N(\theta A^0)]$ implies that $\sum \theta a_r^0 g_r(k_r') f_r(n_r') = \sum \theta a_r^0 g_r(k_r^0) f_r(n_r')$ or $\sum a_r^0 f_r(n_r')[g_r(k_r') - g_r(k_r^0)] = 0$ where $g_r(k_r') \geq g_r(k_r^0)$ for all r. Because (*) holds with $n' = n[\theta A^0, N(\theta A^0)]$, there exists some r for which $n_r' = n_r[\theta A^0, N(\theta A^0)] > 0$ and $k_r^0 \neq k_r^*$. Because (K', N', n') is a VLRE, $n_r' > 0$ implies $k_r' = k_r^*$. Hence there is at least one r with $f_r(n_r') > 0$ and $g_r(k_r') > g_r(k_r^0)$. This contradicts $\sum a_r^0 f_r(n_r')[g_r(k_r') - g_r(k_r^0)] = 0$. Therefore (*) implies $N' > N(\theta A^0)$.

Suppose a^0 is permanently restored and (*) does not hold. We want to show that starting from (K', N', n') the system converges to (K^0, N^0, n^0). We have already shown that if (*) does not hold then $K' = K^0$. The reversion to a^0 from θa^0 is a neutral negative shock. Proposition 5(a) shows that the system converges to a VLRE (K'', N'', n'') such that $K'' = K' = K^0$. The productivity vector for this VLRE is $A'' = A(a^0, K'') = A^0$ and the new VLRE must satisfy $H(A'', N'')/N'' = H(A^0, N'')/N'' = y^*$. This implies $N'' = N^0$. The uniqueness of the solution in (2) gives $n'' = n(A^0, N^0) = n^0$.

Suppose a^0 is permanently restored and (*) does hold. We want to show that starting from (K', N', n') the system converges to a VLRE (K'', N'', n'') with $K'' = K' \neq K^0$ and $N' > N'' \geq N^0$. We have already shown that (*) implies $K' \neq K^0$. Proposition 5(a) shows that the system converges to a VLRE with $K'' = K'$ and $N' > N''$. Thus it suffices to show $N'' \geq N^0$, and to establish conditions under which this inequality is strict. First we show that $N'' \geq N^0$. Define $A'' = A(a^0, K'')$, where $A'' \geq A^0$ due to Proposition 1. If $N'' < N^0$ then $y^* = H(A'', N'')/N'' > H(A'', N^0)/N^0 \geq H(A^0, N^0)/N^0 = y^*$. This contradiction implies $N'' \geq N^0$.

Now suppose $N'' = N^0$ with $n'' \neq n^0$. Because n^0 is feasible in the labor allocation problem for (A'', N'') and the solutions in (2) are unique, $H(A'', N'') = \sum a_r^0 g_r(k_r'')f_r(n_r'') > \sum a_r^0 g_r(k_r'')f_r(n_r^0) \geq \sum a_r^0 g_r(k_r^0)f_r(n_r^0) = H(A^0, N^0)$. This gives $y^* = H(A'', N'')/N'' > H(A^0, N^0)/N^0 = y^*$. This contradiction shows that if $N'' = N^0$ then $n'' = n^0$. We thus have two possibilities: (i) $N'' = N^0$ and $n'' = n^0$ or (ii) $N'' > N^0$ and $n'' \neq n^0$. In either case, because (K^0, N^0, n^0) is a VLRE we have $k_r^0 = k_r^*$ for all r with $n_r^0 > 0$. Proposition 1 implies $k_r'' = k_r' = k_r^*$ for all r with $n_r^0 > 0$.

(i) If $H_N(A^0, N^0) \geq a_r^0 g_r(k_r'')f_r'(0)$ for all r such that $n_r^0 = 0$ then n^0 satisfies the first order conditions for problem (2) with parameters (A'', N^0) because $k_r'' = k_r^0$ for all r with $n_r^0 > 0$. The first order conditions for (2) are sufficient for a solution so $H(A'', N^0)/N^0 = H(A^0, N^0)/N^0 = y^*$. This shows that (K'', N^0, n^0) is a VLRE for the climate a^0. But from Proposition 5(a), the VLRE (K'', N'', n'') is unique. Therefore $N'' = N^0$ and $n'' = n[A(a^0, K''), N^0] = n^0$.

(ii) If $H_N(A^0, N^0) < a_r^0 g_r(k_r'')f_r'(0)$ for some r such that $n_r^0 = 0$ then n^0 does not satisfy the first order conditions for problem (2) with parameters (A'', N^0). Thus $n[A(a^0, K''), N^0] \neq n^0$. Using $N'' \geq N^0$ and the uniqueness of solutions in (2), this gives $H(A'', N'') \geq H(A'', N^0) > \sum a_r^0 g_r(k_r'')f_r(n_r^0) \geq \sum a_r^0 g_r(k_r^0)f_r(n_r^0) = H(A^0, N^0)$. Now suppose $N'' = N^0$. This implies $y^* = H(A'', N'')/N'' = H(A'', N^0)/N^0 > H(A^0, N^0)/N^0 = y^*$, which is a contradiction. Hence $N'' > N^0$. If $n_r'' = 0$ for all r such that $n_r^0 = 0$, there must be at least one s with $0 < n_s^0 < n_s''$. Using $k_r'' = k_r^*$ for all r with $n_r^0 > 0$ and the envelope theorem, this gives $H_N(A'', N'') = a_s^0 g_s(k_s'')f_s'(n_s'') < a_s^0 g_s(k_s^*)f_s'(n_s^0) = H_N(A^0, N^0)$. But then $H_N(A'', N'') < H_N(A^0, N^0) < a_r^0 g_r(k_r'')f_r'(0)$ for some r such that $n_r^0 = 0$, which contradicts the optimality of $n_r'' = 0$ for all r such that $n_r^0 = 0$. Therefore $n_r'' > 0$ for at least one r such that $n_r^0 = 0$.